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The $N_f = 2$ residual mass in perturbative lattice-HQET for an improved determination of $m_b^{\overline{MS}}(m_b^{\overline{MS}})$

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Abstract: We determine to order α^3 the so-called residual mass in the lattice regularisation of the Heavy Quark Effective Theory for $N_f = 2$. Our (gauge-invariant) strategy makes use of Numerical Stochastic Perturbation Theory to compute the static interquark potential where the above mentioned mass term appears as an additive contribution. We discuss how the new coefficient we compute in the expansion of the residual mass can improve the determination of the ($\overline{\mathrm{MS}}$) mass of the b-quark from lattice simulations of the Heavy Quark Effective Theory.

Keywords: Quark Masses and SM Parameters, Heavy Quarks Physics, Lattice QCD.

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Contents

1. Introduction

Given our present computer resources, one can not accommodate the b-quark on the lattice keeping at the same time under control both finite size and finite lattice spacing effects. Despite this fact, the determination of the b-quark mass is one of the most precise results in Lattice Gauge Theory. There are at least four strategies that have been successfully applied to this purpose. A first determination comes from Non-Relativistic QCD [1]. A second one relies on Heavy Quark Effective Theory (HQET) [2]. Both these approaches have to deal with the mass counterterm whose computation we will be concerned with in this paper. In order to avoid this task — which has to be performed in Perturbation Theory (PT) — a third strategy has been devised, which makes use of HQET, but in a Non-Perturbative framework. A direct computation of the mass counterterm is avoided by making use of a finite scaling technique [3]. Finally, a fourth method has been recently introduced which actually avoids HQET and remains within QCD. It again relies on a finite scaling technique, from which it takes the name of Step Scaling Method [4]. A different approach has been followed in [5]. When different strategies are available, an important issue is the comparison of their results. In the quenched approximation, all the previous methods are consistent within errors. At the moment the only unquenched computation is the one in [2]. However the largest source of error, there, comes from the limited knowledge of the perturbative residual mass. This can be strongly improved by the α^3 computation which we are presenting in this paper. One important issue concerning the mass counterterm, whose perturbative computation we are going to present, is that it is a power-divergent quantity. Because of that, there is no continuum limit in PT, and that was the motivation for the approach in [3]. Of course, while this is a conceptually important point, it is not a

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no-go for the first two methods. One has to stick to a finite value of the lattice spacing and to carefully assess the error which is inherent to this procedure. All these considerations set the stage for our computation. Our main task is to make it possible to extend the result in [2] to a higher accuracy in the unquenched case. The quenched determination has already been greatly improved by a previous quenched computation of ours [6] of which the present work is an extension. Having reached the goal of a higher accuracy in the analysis of [2], one can also investigate the status of PT with respect to the fact that one has to stick to a finite lattice spacing.
The structure of the paper is as follows. In the second section we sketch the theoretical framework, while in the third we present our computational strategy. More details about those issues in common with the quenched computation can be found in [6]. The fourth section collects our results together with a brief discussion of the impact of the computation.

2. The residual mass in lattice HQET

The lagrangian density of HQET [7] may be written as

$$\mathcal{L}_{HQET} = \overline{h} D_4 h \,, \tag{2.1}$$

where D_4 is the (gauge) covariant time derivative and h is the bi-spinor describing the only degrees of freedom of the heavy quark that are relevant in this approximation. While the theory is appealing also in force of its simple form, it is well known that there are many subtleties one has to face, many of which have to do with a proper definition of the heavy quark mass itself. This is well evident in the fundamental relation one would like to exploit in order to connect the mass of a physical hadron (M_B) to the HQET expansion mass parameter (m_b) and the (linearly divergent) binding energy ($\mathcal{E}$)

$$M_B = m_b + \mathcal{E} + \mathcal{O}\left(\frac{1}{m_b}\right) . \tag{2.2}$$

At this stage m_b is actually not yet properly defined. Following [2] (and references therein) one can proceed as follows. By matching the QCD propagator to its lattice HQET counterpart one gets a relation involving the pole mass

$$m_b^{\rm pole} = M_B - \mathcal{E} + \delta m + \mathcal{O}\left(\frac{1}{m_b}\right) . \tag{2.3}$$

In this relation a new character has entered the stage: a linearly divergent additive mass counterterm, the so-called residual mass δm, which in lattice regularisation — contrary to dimensional regularisation — is generated by quantum corrections. Our task is now to compute this quantity in Perturbation Theory

$$\delta m = \sum_{n\geq 0} \overline{X_n}\, \alpha_0^{n+1} . \tag{2.4}$$

The pole mass (by writing $m_b^{\rm pole}$ one has in mind the pole mass for the b-quark) can be related to the $\overline{\rm MS}$ mass $\overline{m_b} = m_b^{\overline{\rm MS}}$. The perturbative matching reads

$$\overline{m_b}(\overline{m_b}) = m_b^{\rm pole} \left[1 + \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\overline{m_b})}{\pi} \right)^{n+1} D_n \right] . \tag{2.5}$$

The coefficients D_n with $n \leq 2$ are known [8, 9]. One can now put everything together to get what the authors of [2] call the master equation

$$\overline{m_b}(\overline{m_b}) = \left[M_B - \mathcal{E} + \sum_{n=0}^{\infty} (\alpha_s(\overline{m_b}))^{n+1} \frac{X_n}{a} \right] \left[1 + \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\overline{m_b})}{\pi} \right)^{n+1} D_n \right]. \tag{2.6}$$

After having arrived to this fundamental relation a few comments are in order. Note, first of all, that the last expression asks for both a non-perturbative ($\mathcal{E}$) and a perturbative (δm) computation. As it is well known, $\mathcal{E}$ can be computed as the decay constant of the correlation function of two axial currents. In eq. (2.6) many things are actually taking place, for which the perturbative expansion of the residual mass is responsible. First, δm is in charge of canceling the linear divergence of $\mathcal{E}$. Also, a subtle cancellation of renormalon ambiguities is taking place: the renormalon in the expansion of δm cancels the one in the perturbative relation between the pole and $\overline{\mathrm{MS}}$ masses [10]. It should now be stressed that both things are taking place in Perturbation Theory. Because of that, the cancellation of the linear divergence is incomplete and the continuum limit can not be taken. As for the renormalon ambiguities cancellation, the same coupling must be used in both perturbative expansions present in eq. (2.6). As we have already said, D_2 is known, which means that $\overline{X_2}$ (from which X_2 is computed) has to be computed in order to exploit at best the knowledge coming from continuum Perturbation Theory. $\overline{X_0}$ has been known for a long time, while $\overline{X_1}$ was computed in [11]. As already said, $\overline{X_2}$ was computed in the quenched approximation in [6] and then also in [12]. Our goal is now to compute $\overline{X_2}$ for $N_f = 2$ Wilson fermions, in order to make contact with the unquenched data of [2].

3. The computational strategy

As already said, the computational strategy is just the extension to $N_f = 2$ of the computation in [6]. Consider a gauge invariant loop such as a Wilson (W) or a Polyakov (P) loop. By following [13]'s notation, one can write

$$\langle W \rangle = \exp\left(-\frac{c\,L}{a} \right) W_{\log} . \tag{3.1}$$

In the above formula L is the length of the loop and a the cutoff scale at small distance (of course one can think of the lattice spacing). c is the linear divergence we are interested in. At the time of [13] it was regarded as a mass renormalization of the test particle that one considers if one reads the loop as an effective amplitude. The notation $W_{\log}$ reminds that only logarithmic divergences are left. A first one is the divergence that can be absorbed in the definition of the renormalized coupling. A second logarithmic divergence appears if the contour has an angle; this is usually referred to as the corner divergence. Of course such a problem is there in the case of a Wilson loop and not for a Wilson line. One can now go to Lattice Perturbation Theory (LPT) and compute the static potential via Creutz's ratios

$$V(R) \equiv \lim_{T \to \infty} V_T(R) \qquad V_T(R) \equiv \log\left(\frac{W(R, T-1)}{W(R, T)} \right). \tag{3.2}$$

The corner divergences now disappear because of the ratio, the number of corners being the same in numerator and denominator. As for the coupling, there is a standard way to renormalize it:

$$V(R) = 2\,\delta m + V_{\mathrm{Coul}}(R) \qquad V_{\mathrm{Coul}}(R) \equiv -C_F \frac{\alpha_{\mathrm{V}}(R)}{R}. \tag{3.3}$$

This equation amounts to the definition of the coupling in the potential scheme. Since we are computing in LPT, what we are actually computing in the second formula of eq. (3.3) is the matching between the potential and the lattice couplings. This reads

$$V(R) = 2\,\delta m - \frac{C_F}{R}\left(\alpha_0 + c_1(R)\,\alpha_0^2 + c_2(R)\,\alpha_0^3 + \cdots\right) \tag{3.4}$$

where the relevant coefficients are given by

$$\begin{aligned} c_1(R) &= 2b_0 \log R + 2b_0 \log \frac{\Lambda_V}{\Lambda_0} \\ c_2(R) &= c_1(R)^2 + 2b_1 \log R + 2b_1 \log \frac{\Lambda_V}{\Lambda_0} + \frac{b_2^{(V)} - b_2^{(0)}}{b_0}\,. \end{aligned}$$

The matching coefficients have been written down in terms of the dependence on Λ parameters and coefficients of the β-function. The good message is now that both the matching between the potential and the $\overline{\mathrm{MS}}$ couplings ([14]) and the matching between the $\overline{\mathrm{MS}}$ and the lattice couplings ([15]) are known. This means that one can match the potential and the lattice couplings and as a result the only unknown quantity in eq. (3.4) is δm. This defines our strategy: we computed $V(R)$ in LPT up to the third order and fitted (order by order) δm to make contact with eq. (3.4).

It is worth to comment that the only difference with respect to the procedure in [6] comes from the dependence on N_f of $c_1(R)$ and $c_2(R)$, which in turns comes from the dependence on N_f of Λ_V/Λ_0 and $b_2^{(V)} - b_2^{(0)}$. The final effect of this can be seen[1] in figure (1).

3.1 Unquenching the computation

Our computational tool was Numerical Stochastic Perturbation Theory (NSPT). This a stochastic way of performing Lattice Perturbation Theory which works by expanding the solution of the Langevin equation as a formal power series in the coupling

$$U \to 1 + \sum_{k=1} \beta^{-k/2} U^{(k)}\,.$$

Such a solution has to be plugged in the Langevin equation

$$\frac{\partial}{\partial t} U = \left(-i\nabla S[U] - i\eta\right) U\,,$$

[1] Both in this figure and in the following ones results are plotted in terms of an expansion in α_0. In [6] the plots were in terms of $1/\beta$.

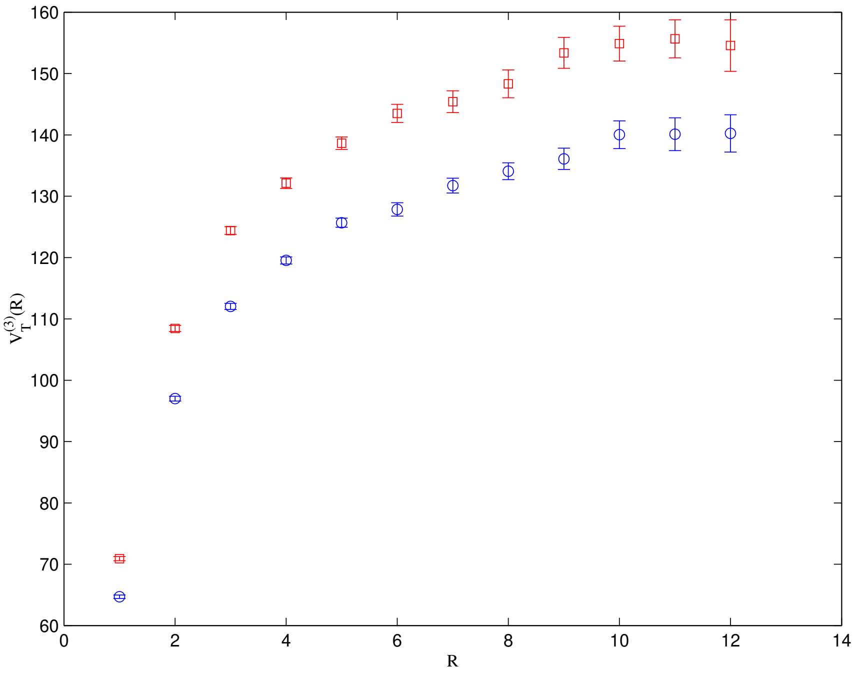

Figure 1: Comparing $V_T^{(3)}(R)$ at $N_f = 0$ (squares) and $N_f = 2$ (circles). This is the third order of the potential with the expansion in terms of α_0.

η being a gaussian noise and ∇ being the Lie derivative on the group that acting on the action produces the classical equation of motion. In the limit of the stochastic time t going to infinite, expectation values with respect to the gaussian (η) noise reconstruct the results of expectation values with respect to the path integral measure [16]. Needless to say, the only difference in computing Wilson loops in the quenched and the unquenched case are the background gauge configurations, which should be computed with a new weight in the functional integral, i.e.

$$e^{-S_G} \mapsto e^{-S_G} \det M = e^{-(S_G - Tr \ln M)} \,,$$

S_G being the Wilson gauge action and M being the lattice Dirac operator. The unquenched version of NSPT requires in the Langevin equation the evaluation of

$$\nabla S_G - \nabla \, \mathrm{Tr} \ln M = \nabla S_G - \mathrm{Tr}((\nabla M) M^{-1}) \,.$$

The second term of the former expression can now be evaluated by a new source of noise ($\langle \xi_i \xi_j \rangle_\xi = \delta_{ij}$)

$$\langle \mathrm{Re}({\xi_l}^\dagger (\nabla M)_{ln} M^{(-1)}_{nk} \xi_k) \rangle_\xi \,,$$

in which a sum is understood on repeated indices. This is an old strategy for unquenched simulations [17], which in NSPT gets greatly simplified by a simple observation. In our method everything should be expanded as a power series in the coupling, which means that also the matrix $M^{(-1)}$ has to be computed as the inverse of

$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)} \,.$$

Now the inverse of this expression reads

$$M^{-1} = {M^{(0)}}^{-1} - \beta^{-1/2} {M^{(0)}}^{-1} M^{(1)} {M^{(0)}}^{-1} + \cdots .$$

The form of the first non trivial order is enough to explain where the good point is. The only inverse to be computed is the zeroth-order, which is diagonal in momentum space and independent of the gauge field. The unquenched version of NSPT was first discussed in [18]. The computation of Wilson loops on which this work relies is its first actual application. It is worth to inspect again figure (1). The $N_f = 0$ data are obtained from a sample of 120 configurations[2], while those at $N_f = 2$ from a sample of 200. As one can see from the errorbars, the statistical fluctuations are practically insensitive to the presence of fermion loops. A reader interested in some extra details on the method can refer to [19]. The computation of Wilson loops is actually only one out of a variety of computations for which the configurations we used were generated for (other goals have to do with full QCD, and in particular mainly with renormalization constants).

4. Results

4.1 δm to third order for $N_f = 2$

To perform the present computation we thermalized 200 configurations on a 32^4 lattice with Wilson gauge action and $N_f = 2$ Wilson fermions, which is the unquenching that is relevant to supplement for an extension of the results in [2]. 100 configurations were produced at $dt = 0.01$ (time step discretization of the Langevin stochastic time), and 100 at $dt = 0.02$. The extrapolation errors at $dt = 0$ are included in the errorbars in figure (1). The mass of both sea quark was set to zero, by exploiting the knowledge of k_{crit}, which — at this order — has been analytically computed in [20, 21]. We computed $W(R,T)$ for all $R, T \leq 16$ up to order α_0^3. Again, a reader interested in the actual results for Wilson loops can refer to [19]. Before presenting our result, we now discuss in some more detail our procedure, together with some comments on errors.

- First of all, statistical errors are associated with the numerical nature of NSPT. These are magnified by the process of extracting the potential, which requires computing a series out of the direct expansions one obtains for the primary quantities (which are the Wilson loops). The statistical errors associated to our measures were computed via the bootstrap technique [22].

- Given the finite lattice size nature of our computation, we can not actually take the $T \to \infty$ limit in eq. (3.2) and it is clear that eq. (3.4) is distorted by lattice artifacts. Given the overall finite size and the presence of two scales (i.e. R and T/R) it follows that there is no simple finite scaling technique one can apply within the constraints of our computation. One issue is of course to take both R and T/R fairly large. One reassuring piece of information concerning lattice artifacts came from the fitting of

[2] Actually these configurations are split into two sets with different discretizations of the time step in the Langevin equation. This is needed in order to perform a continuum stochastic time extrapolation.

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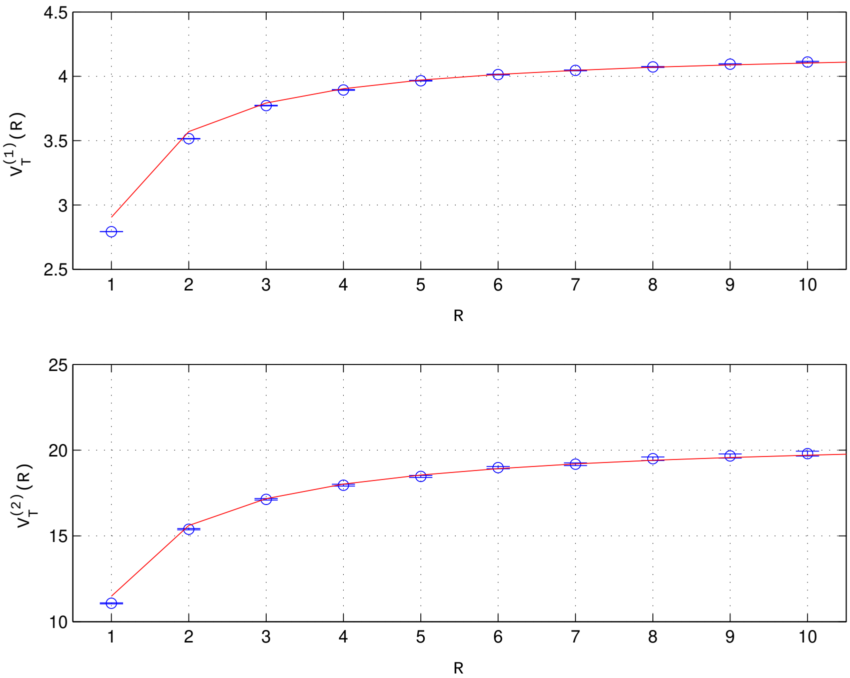

Figure 2: A typical result of the fitting procedure described in the text. The plots refer to first and second order. The solid lines are the expressions of eq. (3.4) once δm has been fitted, while circles are the $V_T(R)$. In this case $T = 14$. Expansions are in terms of α_0.

the constants coefficients entering eq. (3.4) (i.e. C_F, $2b_0 \log \frac{\Lambda_V}{\Lambda_0}$, ...). If one lets them as free parameters, their range of variation within the chosen value of R and T/R (see next point) is only of a few percent.

- We determined the preferred values of the $\overline{X_i}$ by fitting our expansions of $V_T(R)$ to eq. (3.4). In these fits the $\overline{X_i}$ were the only free parameters (i.e. all the other constants were assigned their infinite volume, continuum limit values). The intervals in which fits were performed were such that three conditions were fulfilled: $R \geq 3$, $T \geq 12$ and $T > 2.5\overline{R}$ (we denote by $\overline{R}$ the mean value of R in the fitting interval; the fitting intervals themselves ranged from 3 up to 7 points). It was reassuring to find out that under these conditions good χ^2 values were found.

- As for the values of χ^2 in the fitting intervals we referred to in the previous point, a first observation is that they are quite high for the fit of $\overline{X_0}$: the best values are of order $6, 7$. This is not surprising, given our *fixed lattice artifacts approach*: measurements of $V^{(1)}$ are simply precise enough to clearly detect lattice artifacts effects. For the coefficients $\overline{X_1}$ and $\overline{X_2}$ we quote an error that embraces the range of values for which χ^2 does not exceed the value 2 (again, in the same fitting intervals in which both R and T/R were fairly large).

Figure (2) and figure (3) show an example of our fits at each order.

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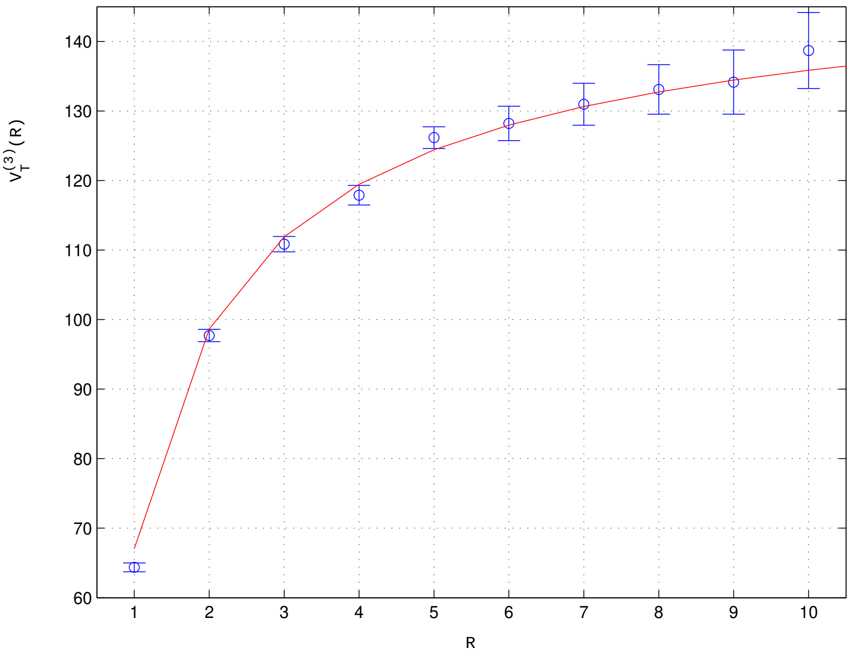

Figure 3: The same as figure (2) for the third order. Again, the expansion is in terms of α_0.

The final results which we obtained are $\overline{X_0} = 2.118(3)$ (to be compared with the analytical $\overline{X_0} = 2.118\ldots$), $\overline{X_1}^{(N_f=2)} = 10.58(4)$ (to compare with [11] $\overline{X_1}^{(N_f=2)} = 10.588\ldots$) and finally

$$\overline{X_2}^{(N_f=2)} = 76.7(6)\,. \tag{4.1}$$

4.2 The impact of the result

The impact of the quenched result of [6] has been known since the Lattice 2000 conference [23]. As for the unquenched case, the authors of [2] have now repeated their analysis for the b-quark mass taking into account the new result for $X_2^{(N_f=2)}$. This results in [24, 25]

$$\overline{m_b}(\overline{m_b})^{(\mathrm{unque})} = (4.21 \pm 0.03 \pm 0.05 \pm 0.04)\;\mathrm{GeV}\,.$$

The last error is the one taking into account the indeterminations in the perturbative matching, on which our result has an effect. It would be twice as large with no knowledge of X_2 (with a central value for the mass of 4.26; see [23]). The new analysis [24] pins down different values for the quenched and unquenched results, even if they are compatible within errors, so that further investigation is needed. As for the finite lattice spacing dependence (remember that in this approach one sticks to finite values of the lattice spacing), it is not dramatic and it gets decreased by including the new term in the matching. On the other hand, the control on renormalon ambiguities seems quite firm.

5. Conclusions

We computed the perturbative expansion of the unquenched ($N_f = 2$) residual mass term in lattice Heavy Quark Effective Theory to order α_0^3. This allows to improve the determination

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of the b-quark mass from (unquenched) lattice simulations of the HQET. Errors appear to be under a fairly good control and the impact on the final result for the quark mass is quite important, even if further investigation is needed if one wants to carefully assess the difference between quenched and unquenched results for the b-quark mass.

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